

A CRITERION OF SOLVABILITY OF THE ELLIPTIC CAUCHY PROBLEM IN A MULTI-DIMENSIONAL CYLINDRICAL DOMAIN

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ABSTRACT. In this paper we consider the Cauchy problem for multidimensional elliptic equations in a cylindrical domain. The method of spectral expansion in eigenfunctions of the Cauchy problem for equations with deviating argument establishes a criterion of the strong solvability of the considered elliptic Cauchy problem. It is shown that the ill-posedness of the elliptic Cauchy problem is equivalent to the existence of an isolated point of the continuous spectrum for a self-adjoint operator with deviating argument.

1. INTRODUCTION

As it is known, the solution of the Cauchy problem for the Laplace equation is unique, but unstable. First of all it should be noted that the existence and uniqueness of its solution is essentially guaranteed by the universal Cauchy - Kovalevskaja theorem, which holds for Elliptic Problems. However, the existence of the solution is guaranteed only in a small. Traditionally the ill-posedness of the elliptic Cauchy problem is determined in relation to its equivalence to Fredholm integral equations of the first kind. The problem of solving the operator equation of the first kind can not be correct, since the operator which is inverse to completely continuous operator is not continuous. The Cauchy problem for the Laplace equation is one of the main examples of ill-posed problems. One can pick up the harmonic functions with arbitrarily small Cauchy data on a piece of the domain boundary, which will be arbitrarily large in the domain (the famous example of Hadamard) [1]. For the formulation of the problem to be correct, it is necessary to narrow down the class of solutions. The stability of a plane problem in the class of bounded solutions firstly was proved by Carleman [2]. From Carleman's results immediately follow estimations characterizing this stability. In the mentioned work Carleman obtained a formula for determinating an analytic function of a complex variable by its values on some piece of the arc. However, this formula is unstable and therefore can not be directly used as an efficient method. The first results related to the construction of an efficient algorithm for solving the problem, best of our knowledge, are published simultaneously in works Carlo Pucci [3] and M.M. Lavrentev [4]. Estimates characterizing the stability of a spatial problem in the class of bounded solutions, were first obtained by M.M. Lavrentev [4] for harmonic functions, given in a straight cylinder and vanishing on the generators. The Cauchy data were given on the base

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of the cylinder. A little later, similar estimates were obtained by S.N. Mergelyan [5] for the functions within a sphere and by M.M. Lavrentev [6] for an arbitrary spatial domain with sufficiently smooth boundary. Around the same time, E.M. Landis [7] obtained estimates characterizing the stability of spatial problem for an arbitrary elliptic equation. The above results laid the foundation for the theory of ill-posed Cauchy problems for elliptic equations. By now this theory has deep development both for the plane, and for the spatial cases, and also for general elliptic equations of high order, etc. Methods of regularization and solutions of ill-posed problems have been proposed in [8]-[14]. In these works the concept of conditional correctness of such problems is introduced and algorithms for constructing their solutions are proposed. In contrast to the presented results, in this paper a new criterion of well-posedness (ill-posedness) initial boundary value problem for a general an unbounded equation is proved. The principal difference of our work from the work of other authors is the application of spectral problems for equations with deviating argument in the study of ill posed boundary value problems. The present work is an extension of results [15]-[17] on the case of more general unbounded operators in a multidimensional cylindrical domain.

2. FORMULATION OF THE PROBLEM AND MAIN RESULTS

Let $\Omega \subset R^n$ be a bounded domain with smooth boundary $\partial\Omega$ and $D = \Omega \times (0, 1)$ is a cylinder. Let \mathcal{L}_x be a self-adjoint unbounded operator in $L_2(\Omega)$ with compact resolvent satisfying the Friedrichs type inequality $(\mathcal{L}_x u, u) \geq \|u\|^2$. By μ_k we denote all eigenvalues (numbered in decreasing order) and by $u_k(x)$, $k \in N$ we denote a complete system of all orthonormal eigenfunctions of the operator \mathcal{L}_x in $L_2(\Omega)$. In D we consider a mixed Cauchy problem for elliptic equations

$$\mathbb{L}u \equiv u_{tt}(t, x) - \mathcal{L}_x u(t, x) = f(t, x), (t, x) \in D, \quad (2.1)$$

with initial conditions

$$u(0, x) = u_t(0, x) = 0, x \in \Omega. \quad (2.2)$$

and for every $t \in (0, 1)$ satisfying a condition $u \in \mathcal{D}(\mathcal{L}_x)$.

Definition 1. *The function $u \in L_2(D)$ we will call a strong solution of the mixed Cauchy problem (2.1), (2.2), if there exists a sequence of functions $u_n \in C^2(\bar{D})$ satisfying conditions (2.2) and (for every $t \in (0, 1)$) belonging to $\mathcal{D}(\mathcal{L}_x)$, such that u_n and Lu_n converge in the norm $L_2(D)$ respectively to u and f .*

In the future, the following eigenvalue problem for an elliptic equation with deviating argument will play an important role.

Find numerical values of λ (eigenvalues), under which the problem for a differential equation with deviating argument

$$\mathbb{L}u \equiv u_{tt}(x, t) - \mathcal{L}_x u(x, t) = \lambda u(x, 1 - t), (x, t) \in D, \quad (2.3)$$

has nonzero solutions (eigenfunctions) satisfying conditions (2.2). Obviously, the equivalent record of equation (2.3) has the form

$$\mathbb{L}Pu \equiv P(u_{tt}(t, x) - \mathcal{L}_x u(t, x)) = \lambda u(t, x), (t, x) \in D,$$

where $Pu(t, x) = u(1 - t, x)$ is a unitary operator.

Theorem 1. *The spectral Cauchy problem (2.3), (2.2) has a complete orthonormal system of eigenfunctions*

$$u_{km}(x, t) = u_k(x) \cdot v_{km}(t), \quad (2.4)$$

where $k, m \in \mathbb{N}$, $v_{km}(t)$ are non-zero solutions of the problem

$$v_{km}''(t) - \mu_k v_{km}(t) = \lambda_{km} v_{km}(1 - t), \quad 0 < t < 1, \quad (2.5)$$

$$v_{km}(0) = v_{km}'(0) = 0, \quad (2.6)$$

and λ_{km} are eigenvalues of problem (2.3), (2.2). In addition for the large k the smallest eigenvalue λ_{k1} has the asymptotic behavior

$$\lambda_{k1} = 4\mu_k e^{-\sqrt{\mu_k}} (1 + o(1)). \quad (2.7)$$

Theorem 2. *A strong solution of the mixed Cauchy problem (2.1) - (2.2) exists if and only if $f(x, t)$ satisfies the inequality*

$$\sum_{k=1}^{\infty} \left| \frac{\tilde{f}_{k1}}{\lambda_{k1}} \right|^2 < \infty, \quad (2.8)$$

where $\tilde{f}_{km} = (f(x, 1 - t), u_{km}(x, t))$.

If condition (2.8) holds, then a solution of problem (2.1) - (2.2) can be written as

$$u(x, t) = \sum_{k=1}^{\infty} \frac{\tilde{f}_{k1}}{\lambda_{k1}} u_{k1}(x, t) + \sum_{k=1}^{\infty} \sum_{m=2}^{\infty} \frac{\tilde{f}_{km}}{\lambda_{km}} u_{km}(x, t). \quad (2.9)$$

By $\tilde{L}_2(D)$ we denote a subspace of $L_2(D)$, spanned by the eigenvectors

$$\{u_{k1}(x, t)\}_{k=p+1}^{\infty}, \quad p \in \mathbb{N}$$

and by $\hat{L}_2(D)$ we denote its orthogonal complement

$$L_2(D) = \tilde{L}_2(D) \oplus \hat{L}_2(D).$$

Theorem 3. *For any $f \in \hat{L}_2(D)$ a solution of the problem (2.1) - (2.2) exists, is unique and belongs to $\hat{L}_2(D)$. This solution is stable and has the form*

$$u(x, t) = \sum_{k=1}^p \frac{\tilde{f}_{k1}}{\lambda_{k1}} u_{k1}(x, t) + \sum_{k=1}^{\infty} \sum_{m=2}^{\infty} \frac{\tilde{f}_{km}}{\lambda_{km}} u_{km}(x, t). \quad (2.10)$$

3. SOME AUXILIARY STATEMENTS

In this section we present some auxiliary results to prove the main results.

Lemma 1. *For each fixed value of the index k the spectral problem (2.5) - (2.6) has a complete orthonormal in $L_2(0, 1)$ system of eigenfunctions $v_{km}(t)$, $m \in \mathbb{N}$, corresponding to the eigenvalues λ_{km} .*

These eigenvalues λ_{km} are roots of the equation

$$\sqrt{\mu_k - \lambda} ch \frac{\sqrt{\mu_k + \lambda}}{2} ch \frac{\sqrt{\mu_k - \lambda}}{2} - \sqrt{\mu_k + \lambda} sh \frac{\sqrt{\mu_k + \lambda}}{2} sh \frac{\sqrt{\mu_k - \lambda}}{2} = 0. \quad (3.1)$$

Proof. Indeed, applying an inverse operator L_C^{-1} to the Cauchy problem (2.5) - (2.6) we arrive at the operator equation

$$v_{km}(t) = \lambda L_C^{-1} P v_{km}(t),$$

where $Pf(t) = f(1-t)$, and a function $\phi(t) = L_C^{-1}f(t)$ is the solution of the Cauchy problem

$$\phi''(t) - \mu_k \phi(t) = f(t), \phi(0) = \phi'(0) = 0, \forall f(t) \in L_2(0, 1).$$

Then for the operator L_C^{-1} we have the representation

$$L_C^{-1}f(t) = \frac{1}{\sqrt{\mu_k}} \int_0^t f(\xi) sh\sqrt{\mu_k}(t-\xi) d\xi, \forall f(t) \in L_2(0, 1). \quad (3.2)$$

Therefore, the adjoint to L_C^{-1} operator has the form

$$(L_C^{-1})^* f(t) = \frac{1}{\sqrt{\mu_k}} \int_t^1 f(\xi) sh\sqrt{\mu_k}(\xi-t) d\xi, \forall f(t) \in L_2(0, 1). \quad (3.3)$$

Taking into account representation (3.2) and (3.3), it is easy to make sure that

$$L_C^{-1}Pf = P(L_C^{-1})^* f.$$

Then the chain of equalities

$$L_C^{-1}Pf = P(L_C^{-1})^* f = P^*(L_C^{-1})^* f = (L_C^{-1}P)^* f, \forall f(t) \in L_2(0, 1),$$

allows us to conclude that the operator $L_C^{-1}P$ is completely continuous self-adjoint Hilbert-Schmidt operator [18]. Therefore for each $k \in \mathbb{N}$, the spectral problem (2.5) - (2.6) has a complete orthonormal system of functions $v_{km}(t)$, $m \in \mathbb{N}$ in $L_2(0, 1)$.

Since \mathcal{L}_x is a self-adjoint positive definite unbounded operator, then all its eigenvalues are real and positive, and system of eigenfunctions forms a complete orthonormal system in $L_2(\Omega)$ [19].

We are looking for eigenfunctions of problem (2.3), (2.2) by means of the Fourier method of separation of variables in the form

$$u_k(x, t) = u_k(x) v(t),$$

where $k \in \mathbb{N}$. Therefore, for determination of unknown function $v(t)$ we get the spectral problem (2.5), (2.6) for an equation with deviating argument.

It is easy to show that the general solution of equation (2.5) has the form

$$v(t) = c_1 ch\sqrt{\mu_k + \lambda} \left(t - \frac{1}{2}\right) + c_2 sh\sqrt{\mu_k - \lambda} \left(t - \frac{1}{2}\right),$$

where c_1 and c_2 are some constants.

Using the initial conditions (2.6), we arrive at the system of linear homogeneous equations concerning these constants. As we know, this system has a nontrivial solution if a determinant of the system

$$\Delta(\lambda) = \begin{vmatrix} ch\frac{\sqrt{\mu_k + \lambda}}{2} & sh\frac{\sqrt{\mu_k - \lambda}}{2} \\ \sqrt{\mu_k + \lambda} sh\frac{\sqrt{\mu_k + \lambda}}{2} & \sqrt{\mu_k - \lambda} ch\frac{\sqrt{\mu_k - \lambda}}{2} \end{vmatrix}$$

is zero. Thus, for determining the parameter λ we get (3.1). The proof of Lemma 1 is complete. \square

Let

$$\varpi_k(\lambda) = \ln \left(\operatorname{cth} \frac{\sqrt{\mu_k + \lambda}}{2} \right) + \ln \left(\operatorname{cth} \frac{\sqrt{\mu_k - \lambda}}{2} \right) - \frac{1}{2} \ln \left(\frac{\mu_k + \lambda}{\mu_k - \lambda} \right) = 0. \quad (3.4)$$

Lemma 2. *There exists a number λ_0 such that for all*

$$0 < \lambda < \lambda_0 < \frac{\mu_k}{4\mu_k + \theta}, \quad k \geq 1, \quad \theta \in (0, 1),$$

the following statements are true:

- 1) *the function $\varpi'_k(\lambda)$ is a constant sign;*
- 2) *for the function $\varpi''_k(\lambda)$ the following inequality holds $|\lambda \mu_k \varpi''_k(\lambda)| < 1$, $k > 1$.*

Proof. By virtue of Lemma 1 we have the real eigenvalues of problem (2.5)-(2.6), that is, real roots λ_{km} of equation (3.1). It is easy to verify that $\lambda_{km} > 0$. Indeed, let us write the asymptotic behavior of the smallest eigenvalues λ_{km} at $k \rightarrow \infty$.

After a nontrivial transformation of equation (3.1), we have

$$\frac{\sqrt{\mu_k + \lambda}}{\sqrt{\mu_k - \lambda}} = \operatorname{cth} \frac{\sqrt{\mu_k + \lambda}}{2} \operatorname{cth} \frac{\sqrt{\mu_k - \lambda}}{2}. \quad (3.5)$$

Assuming $|\lambda| < 1$ and logarithming both sides of equation (3.5), we obtain (3.4). By calculating the derivative $\varpi_k(\lambda)$, we get $\varpi'_k(0) = -\frac{1}{\mu_k}$.

Then the required boundary of monotonicity of $\varpi_k(\lambda)$ can be determined from the relation

$$\varpi'_k(\lambda_0) = \varpi'_k(0) + \varpi''_k(\theta \lambda_0) \lambda_0 < 0,$$

where $\lambda_0 : 0 < \lambda_0 < 1$, and $\theta \in (0, 1)$ is an arbitrary number. Thus, for determining λ_0 we have the condition

$$\lambda_0 \mu_k \varpi''_k(\theta \lambda_0) < 1. \quad (3.6)$$

We write explicitly the second derivative of functions $\varpi_k(\lambda)$:

$$\begin{aligned} \varpi''_k(\lambda) &= \frac{ch\sqrt{\mu_k + \lambda}}{4(\mu_k + \lambda)sh^2\sqrt{\mu_k + \lambda}} + \frac{ch\sqrt{\mu_k - \lambda}}{4(\mu_k - \lambda)sh^2\sqrt{\mu_k - \lambda}} \\ &+ \frac{1}{4\sqrt{(\mu_k + \lambda)^3}sh\sqrt{\mu_k + \lambda}} + \frac{1}{4\sqrt{(\mu_k - \lambda)^3}sh\sqrt{\mu_k - \lambda}} - \frac{2\lambda\mu_k}{(\mu_k^2 - \lambda^2)^2} \\ &= \frac{e^{-3\sqrt{\mu_k + \lambda}}(1 + e^{-2\sqrt{\mu_k + \lambda}})}{2(\mu_k + \lambda)(1 - e^{-2\sqrt{\mu_k + \lambda}})^2} + \frac{e^{-3\sqrt{\mu_k - \lambda}}(1 + e^{-2\sqrt{\mu_k - \lambda}})}{2(\mu_k - \lambda)(1 - e^{-2\sqrt{\mu_k - \lambda}})^2} \\ &+ \frac{e^{-2\sqrt{\mu_k + \lambda}}}{4\sqrt{(\mu_k + \lambda)^3}(1 - e^{-2\sqrt{\mu_k + \lambda}})} + \frac{e^{-2\sqrt{\mu_k - \lambda}}}{4\sqrt{(\mu_k - \lambda)^3}(1 - e^{-2\sqrt{\mu_k - \lambda}})} \\ &- \frac{2\lambda\mu_k}{(\mu_k^2 - \lambda^2)^2}. \end{aligned}$$

Further

$$\begin{aligned} \varpi''_k(\lambda_0\theta) &= \frac{ch\sqrt{\mu_k + \lambda_0\theta}}{4(\mu_k + \lambda_0\theta)sh^2\sqrt{\mu_k + \lambda_0\theta}} + \frac{ch\sqrt{\mu_k - \lambda_0\theta}}{4(\mu_k - \lambda_0\theta)sh^2\sqrt{\mu_k - \lambda_0\theta}} \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{4\sqrt{(\mu_k + \lambda_0\theta)^3} sh\sqrt{\mu_k + \lambda_0\theta}} + \frac{1}{4\sqrt{(\mu_k - \lambda_0\theta)^3} sh\sqrt{\mu_k - \lambda_0\theta}} \\
& - \frac{2\lambda_0\theta\mu_k}{(\mu_k^2 - (\lambda_0\theta)^2)^2}.
\end{aligned}$$

As

$$\frac{2\lambda_0\theta\mu_k}{(\mu_k^2 - (\lambda_0\theta)^2)^2} = \frac{1}{2} \left(\frac{1}{(\mu_k - \lambda_0\theta)^2} - \frac{1}{(\mu_k + \lambda_0\theta)^2} \right) \geq -\frac{1}{(\mu_k + \lambda_0\theta)^2}$$

and

$$\begin{aligned}
& \frac{ch\sqrt{\mu_k + \lambda_0\theta}}{sh^2\sqrt{\mu_k + \lambda_0\theta}} \\
& = \frac{1}{2} \left(\frac{1}{ch\sqrt{\mu_k + \lambda_0\theta} - 1} + \frac{1}{ch\sqrt{\mu_k + \lambda_0\theta} + 1} \right) \\
& \leq \frac{1}{ch\sqrt{\mu_k + \lambda_0\theta} - 1}, \\
& \frac{ch\sqrt{\mu_k - \lambda_0\theta}}{sh^2\sqrt{\mu_k - \lambda_0\theta}} \\
& = \frac{1}{2} \left(\frac{1}{ch\sqrt{\mu_k - \lambda_0\theta} - 1} + \frac{1}{ch\sqrt{\mu_k - \lambda_0\theta} + 1} \right) \\
& \leq \frac{1}{ch\sqrt{\mu_k - \lambda_0\theta} - 1}.
\end{aligned}$$

Then the following inequality is true:

$$\begin{aligned}
& \varpi_k''(\lambda_0\theta) \\
& \leq \frac{1}{4(\mu_k + \lambda_0\theta)(ch\sqrt{\mu_k + \lambda_0\theta} - 1)} + \frac{1}{4(\mu_k - \lambda_0\theta)(ch\sqrt{\mu_k - \lambda_0\theta} - 1)} \\
& + \frac{1}{4\sqrt{(\mu_k + \lambda_0\theta)^3} sh\sqrt{\mu_k + \lambda_0\theta}} \\
& + \frac{1}{4\sqrt{(\mu_k - \lambda_0\theta)^3} sh\sqrt{\mu_k - \lambda_0\theta}} + \frac{1}{(\mu_k + \lambda_0\theta)^2} \\
& \leq \frac{1}{2(\mu_k - \lambda_0\theta)(ch\sqrt{\mu_k - \lambda_0\theta} - 1)} \\
& + \frac{1}{2\sqrt{(\mu_k - \lambda_0\theta)^3} sh\sqrt{\mu_k - \lambda_0\theta}} + \frac{1}{(\mu_k + \lambda_0\theta)^2} \\
& = \frac{e^{-\sqrt{\mu_k - \lambda_0\theta}}}{(\mu_k - \lambda_0\theta)(e^{-2\sqrt{\mu_k - \lambda_0\theta}} - 2e^{-\sqrt{\mu_k - \lambda_0\theta}} + 1)} \\
& + \frac{e^{-\sqrt{\mu_k - \lambda_0\theta}}}{\sqrt{(\mu_k - \lambda_0\theta)^3}(1 - e^{-2\sqrt{\mu_k - \lambda_0\theta}})} + \frac{1}{(\mu_k + \lambda_0\theta)^2}
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{(\mu_k - \lambda_0 \theta)} \left(\frac{e^{-\sqrt{\mu_k - \lambda_0 \theta}}}{(1 - e^{-\sqrt{\mu_k - \lambda_0 \theta}})^2} + \frac{e^{-\sqrt{\mu_k - \lambda_0 \theta}}}{(1 - e^{-2\sqrt{\mu_k - \lambda_0 \theta}})} + 1 \right) \\
&\leq \frac{1}{(\mu_k - \lambda_0 \theta)} \frac{2 + (1 - e^{-\sqrt{\mu_k - \lambda_0 \theta}})^2}{(1 - e^{-\sqrt{\mu_k - \lambda_0 \theta}})^2}.
\end{aligned}$$

Hence

$$\varpi_k''(\lambda_0 \theta) < \frac{1}{(\mu_k - \lambda_0 \theta)} \frac{2 + 1 - 2e^{-\sqrt{\mu_k - \lambda_0 \theta}} + e^{-2\sqrt{\mu_k - \lambda_0 \theta}}}{(1 - e^{-\sqrt{\mu_k - \lambda_0 \theta}})^2} \quad (3.7)$$

Further, for large values k from (3.7) we obtain the validity of the inequality

$$\varpi_k''(\lambda_0 \theta) \leq \frac{4}{\mu_k - \lambda_0 \theta}.$$

Applying the condition (3.6) to the last inequality, we obtain the desired boundary for λ_0 :

$$\lambda_0 < \frac{\mu_k}{4\mu_k + \theta}, \quad k > 1, \quad 0 < \theta < 1.$$

Lemma 2 is proved. \square

Consider now the question of an asymptotic behavior of the eigenvalues of problem (2.5) - (2.6) at large k .

Lemma 3. *An asymptotic behavior of eigenvalues of the problem (2.5)- (2.6), not exceeding λ_0 , for the large values of k has the form (2.7).*

Proof. According to Lemma 2 the monotonic function $\varpi_k(\lambda)$ in the interval $(0, \lambda_0)$ can have only one zero. By the Taylor formula we have

$$\varpi_k(\lambda) = \varpi_k(0) + \frac{\varpi_k'(0)}{1!} \lambda + \frac{\varpi_k''(\theta \lambda)}{2!} \lambda^2 < 0, \quad 0 < \theta < 1.$$

Substituting the calculated values of the function ϖ_k and its derivative ϖ_k' , we get

$$\varpi_k(\lambda) = 2 \ln \left(\operatorname{cth} \frac{\sqrt{\mu_k}}{2} \right) - \frac{\lambda}{\mu_k} + \varpi_k''(\theta \lambda) \frac{\lambda^2}{2}.$$

Then the zero of linear part of the function

$$\mu_k \varpi_k(\lambda) = 2\mu_k \ln \left(\operatorname{cth} \frac{\sqrt{\mu_k}}{2} \right) - \lambda + \frac{\mu_k \lambda^2}{2} \varpi_k''(\theta \lambda)$$

will be

$$\lambda_{k1} = 2\mu_k \ln \left(\operatorname{cth} \frac{\sqrt{\mu_k}}{2} \right) = 2\mu_k \ln \left(\frac{1 + e^{-\sqrt{\mu_k}}}{1 - e^{-\sqrt{\mu_k}}} \right).$$

For sufficiently large values $k \in \mathbb{N}$, considering the asymptotic formulas, λ_{k1} can be written as

$$\lambda_{k1} = 4\mu_k e^{-\sqrt{\mu_k}} (1 + o(1)).$$

Taking into account the result of Lemma 2 on a circle

$$|\lambda| = 4\mu_k e^{-\sqrt{\mu_k}} (1 + \varepsilon),$$

where ε is a greatly small positive number, for sufficiently large $k \geq k_0(\varepsilon)$ it is easy to check the validity of inequality

$$|\varpi_k''(\theta \lambda) \mu_k \lambda^2|_{|\lambda|=4\mu_k e^{-\sqrt{\mu_k}}(1+\varepsilon)}$$

$$\leq C \left| 2\mu_k \ln \left(\frac{1 + e^{-\sqrt{\mu_k}}}{1 - e^{-\sqrt{\mu_k}}} \right) - \lambda \right|_{|\lambda|=4\mu_k e^{-\sqrt{\mu_k}}(1+\varepsilon)}.$$

Then, by Rouché's theorem [20], we have that the quantity of zeros of $\mu_k \varpi_k(\lambda)$ and its linear part coincide and are inside the circle $|\lambda| = 4\mu_k e^{-\sqrt{\mu_k}}(1 + \varepsilon)$. Consequently, the function $\mu_k \varpi_k(\lambda)$ for $0 < \lambda < \lambda_0$ has one zero, the asymptotic behavior is given by formula (2.7). Lemma 3 is proved. \square

4. PROOF OF THE MAIN RESULTS

Proof. (Theorem 1) By $u_k(x)$, $k \in \mathbb{N}$ we have denoted a complete system of orthonormal eigenfunctions of the operator \mathcal{L}_x in $L_2(\Omega)$. By Lemma 1, for each fixed value of the k the spectral problem (2.5) - (2.6) has complete orthonormal system of eigenfunctions $v_{km}(t)$, $m \in \mathbb{N}$ in $L_2(0, 1)$. Then the system (2.4) forms a complete orthogonal system in $L_2(D)$. Consequently, problem (2.3), (2.2) does not have the other eigenvalues and eigenfunctions. Theorem 1 is proved. \square

Proof. (Theorem 2) Let $u(x, t) \in C^2(D)$ be a solution of problem (2.1) - (2.2). Then, by virtue of the completeness and orthonormality of eigenfunctions $u_{km}(x, t)$ of problem (2.3), (2.2), the function $u(x, t)$ in $L_2(D)$ can be expanded in a series [19]

$$u(x, t) = \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} a_{km} u_{km}(x, t), \quad (4.1)$$

where a_{km} are Fourier coefficients of the system $u_{km}(x, t)$. Rewriting equation (2.1) in the form

$$LPu = P(u_{tt}(x, t) - \mathcal{L}_x u(x, t)) = Pf(x, t), \quad (4.2)$$

and substituting the solution of form (4.1) in equation (4.2) according to the ratio

$$P \left(\frac{\partial^2 u_{km}}{\partial t^2}(x, t) - \mathcal{L}_x u_{km}(x, t) \right) = \lambda_{km} u_{km}(x, t),$$

we have

$$a_{km} = \frac{\tilde{f}_{km}}{\lambda_{km}},$$

where $\tilde{f}_{km} = (f(x, 1 - t), u_{km}(x, t))$.

Thus for solutions $u(x, t)$ we obtain the following explicit representation

$$u(x, t) = \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \frac{\tilde{f}_{km}}{\lambda_{km}} u_{km}(x, t). \quad (4.3)$$

Note that the representation (4.3) remains true for any strong solution of problem (2.1)-(2.2). We have obtained this representation under the assumption that the solution of the Cauchy problem (2.1)-(2.2) exists.

The question naturally arises, for what subset of the functions $f \in L_2(D)$ there exists a strong solution?

To answer this question, we represent the formula (4.3) in the form (2.9) from which, by Parseval's equality, it follows

$$\|u\|^2 = \sum_{k=1}^{\infty} \left| \frac{\tilde{f}_{k1}}{\lambda_{k1}} \right|^2 + \sum_{k=1}^{\infty} \sum_{m=2}^{\infty} \left| \frac{\tilde{f}_{km}}{\lambda_{km}} \right|^2. \quad (4.4)$$

By virtue of Lemma 3 we have $\lambda_{km} \geq \frac{1}{4}$, $m > 1$.

Therefore, the right-hand side of equality (4.4) is limited only for those $f(x, t)$, for which the weighted norm (2.8) is limited. This fact proves Theorem 2. \square

Proof. (Theorem 3) Obviously that the operator \mathbb{L} is invariant in $\hat{L}_2(D)$. By Theorem 2, for any $f \in \hat{L}_2(D)$ there exists a unique solution of problem (2.1) - (2.2) and it can be represented in the form (2.10). Therefore, a certain infinite-dimensional space $\hat{L}_2(D)$ is the space of correctness of the Cauchy problem (2.1)-(2.2). Theorem 3 is proved. \square

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